

2. To find the normal equations and to find  $\hat{\mathbf{x}}$ , compute

$$A^T A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

a. The normal equations are  $(A^T A)\mathbf{x} = A^T \mathbf{b}$ :  $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$ .

b. Compute

$$\begin{aligned} \hat{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} -24 \\ -2 \end{bmatrix} \\ &= \frac{1}{56} \begin{bmatrix} 224 \\ 168 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \end{aligned}$$

4. To find the normal equations and to find  $\hat{\mathbf{x}}$ , compute

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

a. The normal equations are  $(A^T A)\mathbf{x} = A^T \mathbf{b}$ :  $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

b. Compute

$$\begin{aligned} \hat{\mathbf{x}} &= (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix} \\ &= \frac{1}{24} \begin{bmatrix} 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

8. From Exercise 4,  $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ , and  $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Since

$$A\hat{\mathbf{x}} - \mathbf{b} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

the least squares error is  $\|A\hat{\mathbf{x}} - \mathbf{b}\| = \sqrt{6}$ .

12. (a) Because the columns  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{a}_3$  of  $A$  are orthogonal, the method of Example 4 may be used to find  $\hat{\mathbf{b}}$ , the orthogonal projection of  $\mathbf{b}$  onto Col  $A$ :

$$\begin{aligned}\hat{\mathbf{b}} &= \frac{\mathbf{b} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 + \frac{\mathbf{b} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2} \mathbf{a}_2 + \frac{\mathbf{b} \cdot \mathbf{a}_3}{\mathbf{a}_3 \cdot \mathbf{a}_3} \mathbf{a}_3 = \frac{1}{3} \mathbf{a}_1 + \frac{14}{3} \mathbf{a}_2 + \left(-\frac{5}{3}\right) \mathbf{a}_3 \\ &= \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}\end{aligned}$$

- (b) The vector  $\hat{\mathbf{x}}$  contains the weights which must be placed on  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$  to produce  $\hat{\mathbf{b}}$ . These

weights are easily read from the above equation, so  $\hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$ .

**24.** Since in this case  $A^T A = I$ , the normal equations give  $\hat{\mathbf{x}} = A^T \mathbf{b}$ .

**25.** The normal equations are  $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ , whose solution is the set of all  $(x, y)$  such that  $x + y = 3$ .

The solutions correspond to the points on the line midway between the lines  $x + y = 2$  and  $x + y = 4$ .